A Study on FEA of Torsional Vibration in Geared Shafts

A Project Report
Submitted in partial fulfillment for the award of the degree
Of

BACHELOR OF TECHNOLOGY
IN
MECHANICAL ENGINEERING

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2010

CERTIFICATE

This is to certify that the thesis entitled, "A Study on FEA of Torsional Vibration in Geared Shafts" submitted by Sri Debdeep Ray in partial fulfillments for the requirements for the award of Bachelor of Technology Degree in Mechanical Engineering at National Institute of Technology, Rourkela (Deemed University) is an authentic work carried out by him under my supervision and guidance.

To the best of my knowledge, the matter embodied in the thesis has not been submitted to any other University / Institute for the award of any Degree or Diploma.

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ACKNOWLEDGEMENT

I place on record and warmly acknowledge the continuous encouragement, Invaluable supervision, timely suggestions and inspired guidance offered by our guide **Prof. N.Kavi,** Professor, Department of Mechanical Engineering, National Institute of Technology, Rourkela, in bringing this report to a successful completion. An erudite teacher and a magnificent person I consider myself fortunate to have worked under his supervision.

I would like to express my gratitude to **Prof. R.K. Sahoo** (Head of the Department) and **Prof. K.P. Maity** (Project Coordinator) for their valuable suggestions and encouragements at various stages of the work. I am also thankful to all staff & members of Department of Mechanical Engineering, NIT Rourkela.

Finally I extend my gratefulness to one and all who are directly or indirectly involved in the successful completion of this project work.

Debdeep Ray (10603016)

ABSTRACT

Finite element analysis is a sophisticated technology based on the principle of discretization and numerical approximation to solve scientific and engineering problems. In this methodology any structure under consideration is discretized into small geometric shapes and the material properties are analyzed over these small elements. The purpose of this project is to study the simple approach of analyzing the torsional vibration in a branched geared system. The global mass matrix and stiffness matrix of the geared system are obtained using the finite element method (FEM) by aggregating the property matrices of the elements discretized in the shafts present in the torsional system. The masses are lumped in the form of rotors in the geared system. Then equations of motion of the whole vibrating system are defined. A finite assemblage of discrete continuous elements leads to the formulation of the associated eigenvalue equation in the configuration of matrices provides the natural frequencies and the mode shapes of the system. Holzer's method is also discussed to find the natural frequencies of branched geared systems. And thence the results obtained using Finite Element Method are compared with those obtained by the numerical method as devised by 'Holzer'.

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Chapter 1

INTRODUCTION

1.1 Finite Element Method

Finite element analysis is based on the principle of discretization and numerical approximation to solve scientific and engineering problems. In this method, a complex region defining a continuum is discretized into simple geometric shapes called the finite elements. The material properties and the governing relationships are considered over these elements and are expressed in terms of unknown elements at the corners. An assembly process duly considering the loading and constraints results in a set of equations. Solution of these equations gives the approximate behaviour of the continuum. The application of this method ranges from deformation and stress analysis of automotives, aircrafts, buildings, bridge structures to field analysis of other flow problems. With the advent of new computer technologies and CAD systems complex problems can be modeled with relative ease. Several alternative configurations can be tested on a computer before the first prototype is built. All these above suggests that we need to keep pace with these developments by understanding the basic theory, modeling techniques and computational aspects of finite element analysis.

1.1.3 Fundamental Concepts

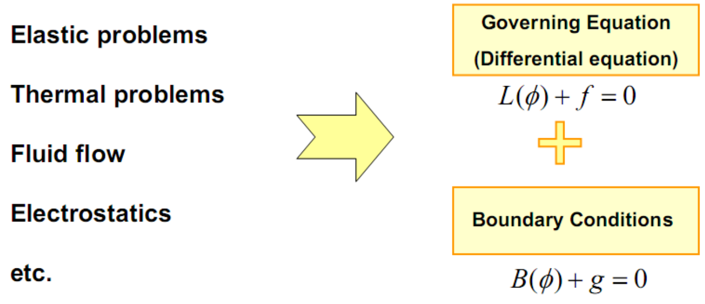


Fig. (1.1)

It is very difficult to make the alegebraic equations for the entire domain

- Divide the domain into a number of small, simple elements
- A field quantity is interpolated by a polynomial over an element
- Adjacent elements share the DOF at connecting nodes

$$\underset{\text{Property}}{[\mathbf{K}]}\underset{\text{Behavior}}{\{\mathbf{u}\}} = \underset{\text{Action}}{\{\mathbf{F}\}} \quad \Rightarrow \quad \{\mathbf{u}\} = [\mathbf{K}]^{-1}\{\mathbf{F}\}$$

1.1.3 <u>FEM Advantages</u>

- Can readily handle complex geometry
    - The heart and power of FEM
- Can handle a wide variety of engineering problems
    - Solid and Fluid Mechanics
    - Dynamics
    - Heat Problems
    - Electrostatic Problems
- Can handle complex structures
    - Indeterminate structures can be solved
- Can handle complex loading
    - Nodal load (point load)
    - Element load (pressure, thermal, inertial forces)
    - Time or frequency dependent loading

1.2 Torsional Vibration

1.2.1 Vibration

Vibration refers to mechanical oscillations about an equilibrium point. The oscillations may be periodic such as the motion of a pendulum or random such as the movement of a tire on a gravel road. Vibration is occasionally "desirable". For example the motion of a tuning fork, the reed in a woodwind instrument or harmonica, or the cone of a loudspeaker is desirable vibration, necessary for the correct functioning of the various devices. More often, vibration is undesirable, wasting energy and creating unwanted sound – noise. For example, the vibrational motions of engines, electric motors, or any mechanical device in operation are typically unwanted. Such vibrations can be caused by imbalances in the rotating parts, uneven friction, the meshing of gear teeth, etc. Careful designs usually minimize unwanted vibrations.

Free vibration occurs when a mechanical system is set off with an initial input and then allowed to vibrate freely. Examples of this type of vibration are pulling a child back on a swing and then letting go or hitting a tuning fork and letting it ring. The mechanical system will then vibrate at one or more of its "natural frequencies" and damp down to zero.

Forced vibration is when an alternating force or motion is applied to a mechanical system. Examples of this type of vibration include a shaking washing machine due to an imbalance, transportation vibration (caused by truck engine, springs, road, etc), or the vibration of a building during an earthquake. In forced vibration the frequency of the vibration is the frequency of the force or motion applied, with order of magnitude being dependent on the actual mechanical system.

1.2.2 Torsional Vibration

Torsional vibration is angular vibration of an object—commonly a shaft along its axis of rotation. Torsional vibration is often a concern in power transmission systems using rotating shafts or couplings where it can cause failures if not controlled. In ideal power transmission systems using rotating parts the torques applied or reacted are "smooth" leading to constant speeds. In reality this is not the case. The torques generated may not be smooth (e.g., internal combustion engines) or the component being driven may not react to the torque smoothly (e.g., reciprocating compressors). Also, the components transmitting the torque can generate non-smooth or alternating torques (e.g., worn gears, misaligned shafts). Because the components in power transmission systems are not infinitely stiff these alternating torques cause vibration along the axis of rotation.

Torsional vibrations may result in shafts from following forcings:

- Inertia forces of reciprocating mechanisms (such as pistons in Internal Combustion engines)
- Impulsive loads occurring during a normal machine cycle (e.g. during operations of a punch press)
- Shock loads applied to electrical machineries (such as a generator line fault followed by fault removal and automatic closure)
- Torques related to gear tooth meshing frequencies, turbine blade passing frequencies, etc.

For machines having massive rotors and flexible shafts (where system natural frequencies of torsional vibrations may be close to, or within, the source frequency range during normal operation) torsional vibrations constitute a potential design problem area.

In such cases designers should ensure the accurate prediction of machine torsional frequencies and frequencies of any of the torsional load fluctuations should not coincide with torsional natural frequencies. Hence, determination of torsional natural frequencies of a dynamic system is very important.

1.3 Geared Systems

A gear is a rotating machine part having cut teeth, or cogs, which mesh with another toothed part in order to transmit torque. Two or more gears working in tandem are called a transmission and can produce a mechanical advantage through a gear ratio and thus may be considered a simple machine. Geared devices can change the speed, magnitude, and direction of a power source. The most common situation is for a gear to mesh with another gear, however a gear can also mesh a non-rotating toothed part, called a rack, thereby producing translation instead of rotation. The gears in a transmission are analogous to the wheels in a pulley. An advantage of gears is that the teeth of a gear prevent slipping. When two gears of unequal number of teeth are combined a mechanical advantage is produced, with both the rotational speeds and the torques of the two gears differing in a simple relationship. In transmissions which offer multiple gear ratios, such as bicycles and cars, the term gear, as in first gear, refers to a gear ratio rather than an actual physical gear. The term is used to describe similar devices even when gear ratio is continuous rather than discrete, or when the device does not actually contain any gears, as in a continuously variable transmission.

CHAPTER 2

LITERATURE SURVEY

2.1 Rotating machines

Rotating machines like steam turbines, compressors, generators most probably develop excessive dynamic stresses, when they run at speeds near their natural frequencies in torsional vibration. Continuous operation of the machinery under such conditions can lead to premature fatigue failure of system components. One of the major obstacles in the measurement and subsequent detection of torsional vibration in a machine is that torsional oscillations cannot be detected without special equipment. However, prediction of torsional natural frequencies of a system and consequent design changes that avoid the torsional natural frequencies from occurring in the operating speed range of a machine is necessary. Many a time, torsional vibration produces stress reversals causing metal fatigue and gear tooth impact forces. All rigid bodies such as flywheel, inner and outer parts of a flexible coupling, a turning disk, can be considered as rigid disks, whose mass moments of inertia can be found out easily. Flexible couplings and thin shafts, whose polar mass moments of inertia are small, can be considered as massless shafts, whose torsional stiffness can be determined easily. Where the shaft diameter is large and its polar mass moment of inertia cannot be neglected as in the case of steam turbine or generator shaft, we can either consider a large number of sections and lump the inertia of each section as a rigid as a rigid disk while retaining a large number of stations and lump the inertia of each station as a rigid disk while keeping the elasticity of the shaft as in massless torsional shafts, or alternatively we can consider a distributed inertia and stiffness of the shaft between sections, where the shaft diameter is large and that its inertia cannot be ignored. Hence the system reduces to either several rigid disks connected by massless elastic shafts or distributed mass and elastic shafts. If one part of the system is coupled to another part through gears, the system interias and stiffnesses should be reduced to one reference speed. [16]

So we see that, the torsional characteristics of a system greatly depend on the stiffness and inertia in the train. While some properties of the system can be changed, generally the system inertia cannot be altered as required. Considering the case of a pump, whose inertia properties are depend upon the dimensions of its impellers, shafts, driving motors etc. Now, a change in geometries and overall sizes to effect torsional characteristics may faint the consideration of factors like the pump hydraulics and lateral vibrations. Besides, the selection of the driver, which is primarily based on the power and load requirements, can hardly be fixed based on torsional characteristics. The typical engineering objectives of torsional vibration analysis are listed below [3]:

1. Predicting the torsional natural frequencies of the system.
2. Calculating the effect of the natural frequencies and vibration amplitudes of changing one or more design parameters (for e.g. "sensitivity analysis").
3. Determining vibration amplitudes and peak torque under steady-state torsional excitation.
4. Computing the dynamic torque and gear tooth loads under transient conditions (for e.g., during machine startup).

2.2 Gear Dynamics

Gears are comprehensively used for power transmission in many engineering machines like vehicles and industrial devices. The dynamic behavior of gears has a great influence on noise and vibration of the system that gears drive. Hence, design of gear systems affects the performance of such machines significantly. Many papers have been published in the past on the effect of dynamics of gears on the response of the system [4-6]. It seems that most of the works are on vibrations of gears caused by backlash, alteration of tooth profile and eccentricities. The backlash detection and its influence in geared shafts has been investigated by N Sarkar et al [7]. Ambili and A Fregolent [8] found out the modal parameters of spur gear system using Harmonic Balance Method. Studying of design of compact spur gears including the effects of tooth stress and dynamic response was carried by PH Lin et al [5]. Non-linear behavior of gear system with backlash and varying stiffness was studied by S. Nastivas & S. Theodassiadas [9]. The study of modal analysis of compliant multi-body geared systems done by H. Vinayak and R. Singh [10]. The effect of time-variant meshing stiffness and non-uniform gear speed on dynamic performance using Finite Elements was analysed by Y. Wang et al [11]. A gear system can be seen as a system of rotors interacting with one another dynamically. Therefore, linear as well as gyroscopic whirling phenomena are expected to exist in addition to vibrations caused by tooth, bearings and shaft interactions. The rotor effect is considered negligible in the gear systems with bearing supports in both ends because design can be made to minimize lateral deflections; however can be a significant effect in overhung gears such as seen in hypoid gear systems. A dynamic model for geared multi-body system containing gear, bar and shaft was proposed and a new gear element, particularly developed based on a Finite element theory by Yong Wang et al. [12]

CHAPTER 3

METHODS TO FIND NATURAL FREQUENCIES

The various methods [13] are:

1. Dunkerly's Equation

   Gives good results if damping is negligible and the frequencies of the harmonics are much higher than the fundamentals.

2. Rayleigh Method

   Here, the dynamic mode shape or modal vector is assumed to estimate the natural frequency.

3. Holzer's Method

   This method assumes a trial frequency. A solution is found when the assumed frequency satisfies the constraints of the problem using a systematic tabulation of frequency equations.

4. Transfer Matrix

   Here, the concept of state vector and transfer matrices are applied to the technique of Holzer's Method.

5. Finite Element Method

   An eigenvalue equation is formed using elemental stiffness matrices (using displacement or direct stiffness method) and mass matrices (using Lagrange's Equations.

For this project the Finite Element Method is used mainly to determine the natural frequencies and mode shapes and then the results hence obtained were compared using those calculated using Holzer's Method. So now, the Holzer's Method and FEM are explained in detail.

3.1 Holzer's Method

3.1.1 For Straight Systems

For Straight Systems described in [13], Holzer method is basicaly a systematic tabulation of the frequency equations of the system. The method has general applications, spanning systems with rectilinear and angular motions, damped or undamped, un-branched or branched. Here, a trial frequency is assumed. When the assumed frequency satisfies the constraints, the solution is reached. Normally, it requires several trials. The iterations also gives us the mode shapes. To explain consider a 3 rotor straight system,

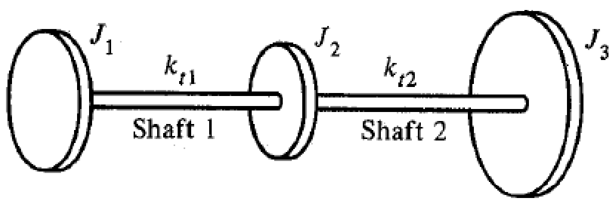


Fig. (3.1)

By Newton's Second Law,

$$\begin{aligned} J_1\ddot{\theta} &= -k_{t1}(\theta_1 - \theta_2) \\ J_2\ddot{\theta} &= -k_{t1}(\theta_2 - \theta_1) - k_{t2}(\theta_2 - \theta_3) \\ J_3\ddot{\theta} &= -k_{t2}(\theta_3 - \theta_2) \end{aligned} \qquad (1)$$

Now, since the motions are harmonic at principal mode of vibration,

$$\theta_i = \Theta_i \sin \omega t \qquad (2)$$

Substituting (2) in (1),

$$
\begin{aligned}
-\omega^2 J_1 \Theta &= -k_{t1}(\Theta_1 - \Theta_2) \\
-\omega^2 J_2 \Theta &= -k_{t1}(\Theta_2 - \Theta_1) - k_{t2}(\Theta_2 - \Theta_3) \\
-\omega^2 J_3 \Theta &= -k_{t2}(\Theta_3 - \Theta_2)
\end{aligned} \tag{3}
$$

Summing up the equations,

$$\sum_{i=1}^{3} J_i \Theta_i \omega^2 = 0 \tag{4}$$

And for n rotor system,

$$\sum_{i=1}^{n} J_i \Theta_i \omega^2 = 0 \tag{5}$$

Generalized for n disk system to find angular displacement,

$$\Theta_j = \Theta_{j-1} - \frac{\omega^2}{k_{t(j-1)}} \sum_{i=1}^{j-1} J_i \Theta_i \tag{6}$$

Hence assuming a initial trial frequency ω and θ, and then using the equations (5) and (6), repeatedly gives us the natural frequencies when (5) is satisfied. The residual torque in (5) represents the torque applied to the last disk, which is a condition of steady-state forced vibration. The amplitudes Θ give the mode shapes.

3.1.2 <u>For Branched Systems</u>

In previous sections we considered straight systems, but in several mechanical systems like sheep propulsion systems, turbine systems, etc. they may employ one or more drivers driving one or more driven members. Consider a branched system as described in [14] is shown in fig. (2) , with three branches A, B and C, denoted by n_a,n_b and n_c. The overall transfer matrices are as follows,

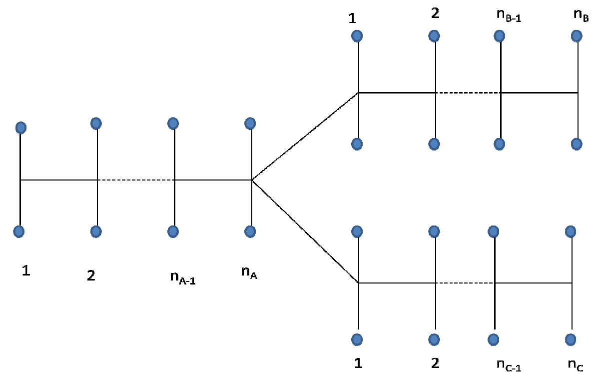


Fig. (3.2)

$$\begin{aligned}\{S\}_{nA}^{R} &= [A]\{S\}_{Ao} \\ \{S\}_{nB}^{R} &= [B]\{S\}_{Bo} \\ \{S\}_{nC}^{R} &= [C]\{S\}_{Co}\end{aligned} \tag{7}$$

At the branch point following points are satisfied,

$$\begin{aligned}\theta_{nA} &= \theta_{Bo} = \theta_{Co} \\ T_{nA} &= T_{Bo} = T_{Co}\end{aligned} \tag{8}$$

Using the condition of free end 1 of A with θ_{Ao}=1, and from (7),

$$\begin{Bmatrix}\theta \\ T\end{Bmatrix} = \begin{pmatrix}A_{11} & A_{12} \\ A_{21} & A_{22}\end{pmatrix}\begin{Bmatrix}1 \\ 0\end{Bmatrix} \tag{9}$$

Hence,

$$
\begin{aligned}
T_{nA}^{R} &= A_{21} \\
\theta_{nA}^{R} &= A_{11}
\end{aligned}
\tag{10}
$$

Using (10) and (8) in (7),

$$
\begin{Bmatrix} \theta \\ 0 \end{Bmatrix}_{nB}^{R} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{Bmatrix} A_{11} \\ T \end{Bmatrix}_{Bo} \tag{11}
$$

Therefore,

$$
T_{Bo} = -\frac{B_{21}A_{11}}{B_{22}} \tag{12}
$$

Using (10) and (12) in (8),

$$
T_{Co} = A_{21} + \frac{B_{21}A_{11}}{B_{22}} \tag{13}
$$

The third equation in (7) can now be written as

$$
\begin{Bmatrix} \theta \\ 0 \end{Bmatrix}_{nC}^{R} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{Bmatrix} A_{11} \\ A_{21} + \dfrac{B_{21}A_{11}}{B_{22}} \end{Bmatrix} \tag{14}
$$

The frequency equation is therefore,

$$
C_{21}A_{11} + C_{22}A_{21} + \frac{C_{22}B_{21}A_{11}}{B_{22}} = 0 \tag{15}
$$

3.2 Finite Element Method

A complicated structure is regarded as a finite assemblage of discrete continuous elements. The basic purpose of the modeling is to get the component equations of motion in the configuration of a large matrix. The FEM provides a systematic way of obtaining these equations, with ideally no restrictions on the system geometry, in a form suitable for computer effectuation. Defining a Finite Element model means, [17]

- Setting up the spatial locations of "nodal points", where the angular displacements are to be determined and external torques applied.
- The continuity of shaft properties is then used to tie the shaft elements at appropriate node points.
- Point inertias are added at nodal points where necessary to consummate the model.
- The element mass and stiffness matrices are assembled to obtain the global mass and stiffness matrices.

3.2.1 Geared Elements (Fundamental Information)

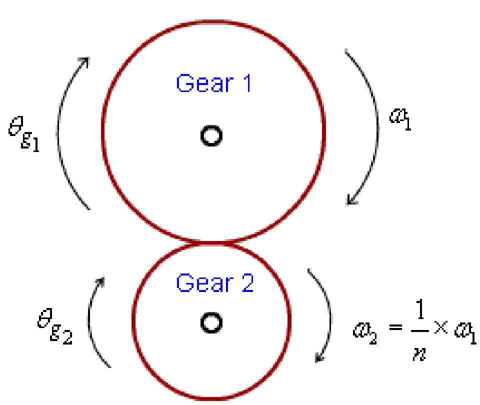


Fig. (3.3)

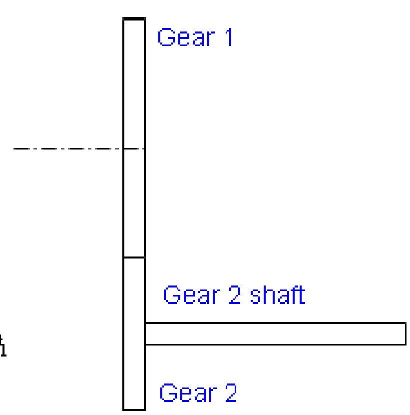


Fig. (3.4)

Figures 1 & 2 shows gear element pair elements. The counter clockwise direction has been taken as positive direction and for all angular displacements and positive convention has been taken as it is practice and convenient in finite element formulations. The gear ratio, n, (which is inverse of speed ratio) is defined as [16],

$$n = \frac{\omega_1}{\omega_2} \tag{16}$$

where ω_1 and ω_2 are the angular speed of driver and driven shafts. For no slip condition from Fig. (3), we have

$$\theta_{g2} = -n\theta_{g1} \tag{17}$$

where θ_{g1} and θ_{g2} are angular displacements of gear 1 and 2, respectively. Since θ_{g2} is defined in terms of θ_{g1} we can eliminate θ_{g2} from the state vector of the system.

The stiffness matrix of shaft element connected to gear 2 will have to be modified, since the angular deflection of the left hand side of that element is now $\frac{-1}{n}\theta_{g1}$ (see Fig. (5))

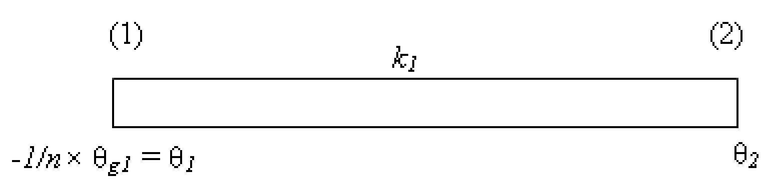


Fig. (3.5)

The potential energy of the shaft element is [15],

$$U_1 = \frac{1}{2}k_1(-\theta_1 + \theta_2)^2 = \frac{1}{2}k_1\left(\frac{\theta_{g1}}{n} + \theta_2\right)^2 \tag{18}$$

The external work done is give as

$$W_1 = -f_1\theta_1 - f_2\theta_2 = f_1\frac{\theta_{g2}}{n} - f_2\theta_2 \tag{19}$$

On applying Ritz method, we get

$$\frac{\partial(U_1+W_1)}{\partial\theta_{g1}} = \frac{k_1}{n}\left(\frac{\theta_{g1}}{n}+\theta_2\right)+\frac{f_1}{n}=0 \tag{20}$$

and

$$\frac{\partial(U_1+W_1)}{\partial\theta_2} = k_1\left(\frac{\theta_{g1}}{n}+\theta_2\right)-f_2=0 \tag{21}$$

3.2.2 Geared Shaft Systems

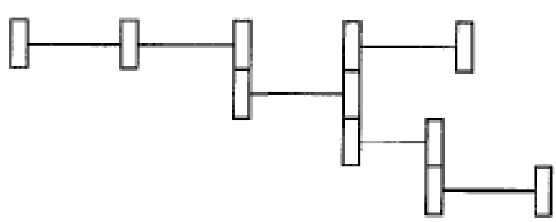


Fig. (3.6)

The general geared shaft system, as shown in Fig. (6), consists of rotating inertias, I_i coupled by elastic shafts, K_j. In the following treatment, it is assumed that the gear assembly is torsionally rigid, i.e., tooth and gearbox flexibility and gear backlash are neglected. [1]

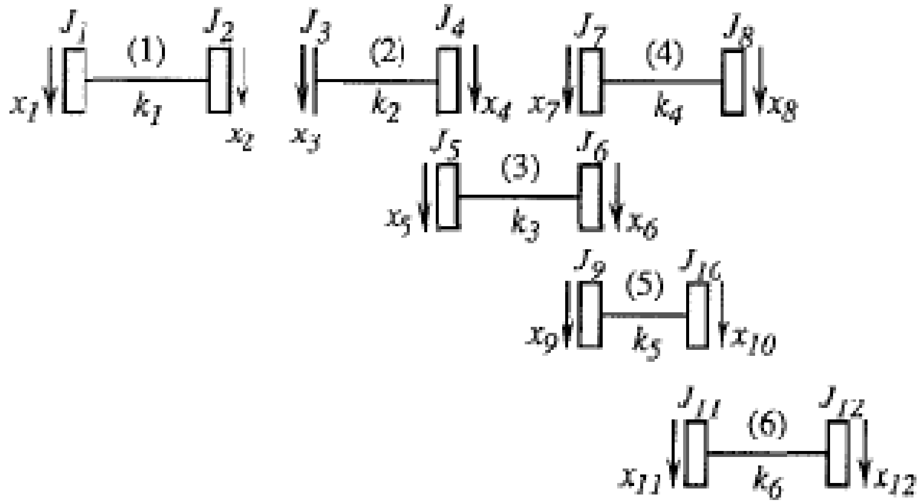


Fig. (3.7)

The finite element model of the system is shown in Fig. 7. Each element consists of two rotating inertias coupled by an elastic shaft. Generally, the stiffness matrix and the inertia matrix of the *i*th element can be shown to be [1],

$$K_i^e = \begin{bmatrix} K_i & -K_i \\ -K_i & K_i \end{bmatrix} \tag{22}$$

$$I_i^e = \begin{bmatrix} I_{2i-1} & 0 \\ 0 & I_{21} \end{bmatrix} \tag{23}$$

The displacement-based finite element equilibrium equations of free-free torsional vibration of the system can be shown to be,

$$\begin{aligned} &[I]\{\ddot{\theta}\}+[K]\{\theta\}=\{0\} \\ &where,\{\theta\}=\{\theta_1,\theta_1,\theta_1,\ldots\ldots\}^T \end{aligned} \tag{24}$$

The above equation is for free vibration. For forced Vibration,

$$[I]\{\ddot{\theta}\}+[K]\{\theta\}=\{F\} \tag{25}$$

3.2.3 Derivation of Property Matrices due to Meshing of Gears [2]

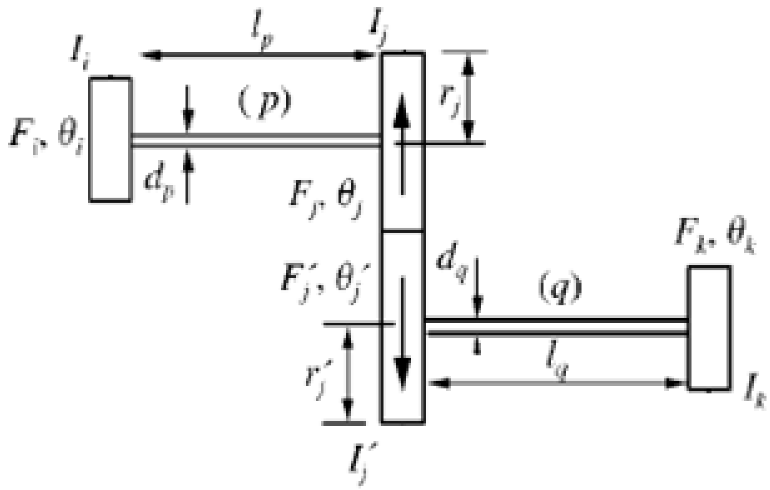


Fig. (3.8)

Fig. (8) shows single branched gear system, where shaft p and shaft q are connected by meshing of 2 gears with pitch circle radii r_j and $r^{'}_{\mathrm{j}}$ respectively. The equations of motion for the two-shaft elements containing the elemental stiffness and mass matrices are given by,

$$\begin{bmatrix} I_i & 0 \\ 0 & I_j \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_i \\ \ddot{\theta}_j \end{Bmatrix}+\begin{bmatrix} k_p & -k_p \\ -k_p & k_p \end{bmatrix}\begin{Bmatrix} \theta_i \\ \theta_j \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{26}$$

$$\begin{bmatrix} I_i^{'} & 0 \\ 0 & I_k \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_j \\ \ddot{\theta}_k \end{Bmatrix}+\begin{bmatrix} k_q & -k_g \\ -k_q & k_q \end{bmatrix}\begin{Bmatrix} \theta_i^{'} \\ \theta_j^{'} \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{27}$$

The notations with subscripts $i, j, j^{'}$ and k, respectively, represent the associated parameters of gears $i, j, j^{'}$ and k. Similarly, those with subscripts p and q denote the associated parameters of shafts p and q respectively. For angular displacement compatibility,

$$r_j\theta_j = -r_j^{'}\theta_j^{'} \tag{28}$$

$$\theta_j^{'} = -\frac{r_j}{r_j^{'}}\theta_j = -n\theta_j \tag{29}$$

Here, n represents the speed ratio or gear ratio between shaft p and q.

$$\begin{Bmatrix} \theta_j^{'} \\ \theta_k \end{Bmatrix} = \begin{bmatrix} -n & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \theta_j \\ \theta_k \end{Bmatrix} \tag{30}$$

Similarly for angular acceleration,

$$\begin{Bmatrix} \ddot{\theta}_j^{'} \\ \theta_k \end{Bmatrix} = \begin{bmatrix} -n & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_j \\ \ddot{\theta}_k \end{Bmatrix} \tag{31}$$

Using (30) and (31) in (27),

$$[m_e]\{\ddot{\theta}\} + [k_e]\{\ddot{\theta}\} = \{0\} \tag{32}$$

Where,

$$[m_e] = \begin{bmatrix} n^2 I_j^{'} & 0 \\ 0 & I_k \end{bmatrix}, [k_e] = \begin{bmatrix} n^2 k_q & nk_q \\ nk_q & k_q \end{bmatrix} \tag{33}$$

It is seen that the angular displacement θ_j' is eliminated in equation (32). Also, the elemental mass matrix [m_e] and stiffness matrix [k_e] driven shaft q, defined by equations (33) will take the same form as those of the driving shaft p.

So now, to find the eigen values and eigen vectors,

$$[M]\{\ddot{\theta}\}+[K]\{\theta\}=0 \tag{34}$$

Where, [M] is the global mass matrix and [K] is the global stiffness matrix.

Let,

$$\{\theta\}=\{\Theta\}e^{i\omega t} \tag{35}$$

Where, $\{\Theta\}$ is the amplitude of $\{\theta\}$ and ω is the natural frequency.

Substituting (35) in (34) gives,

$$([K]-\omega^2[M])\{\Theta\}=0 \tag{36}$$

This is an eigenvalue equation which will give us the natural frequencies and then correspondingly the mode shapes of the vibrating system.

CHAPTER 4

PROBLEM DESCRIPTION AND RESULTS

In the project three types of branched systems are analyzed, i.e. the natural frequencies and the corresponding mode shapes are determined. A comprehensive Matlab code [18] is written to input the dimensions and geometry of the gear system, which first forms the elemental stiffness and mass matrices and then determines the global matrices. The eigenvalue equation hence found is solved to calculate the eigen values (natural frequencies) and eigen vectors (mode shapes). The natural frequencies found using finite element method are then compared with the values calculated using Holzer's Method.

4.1 Single Branched Gear System

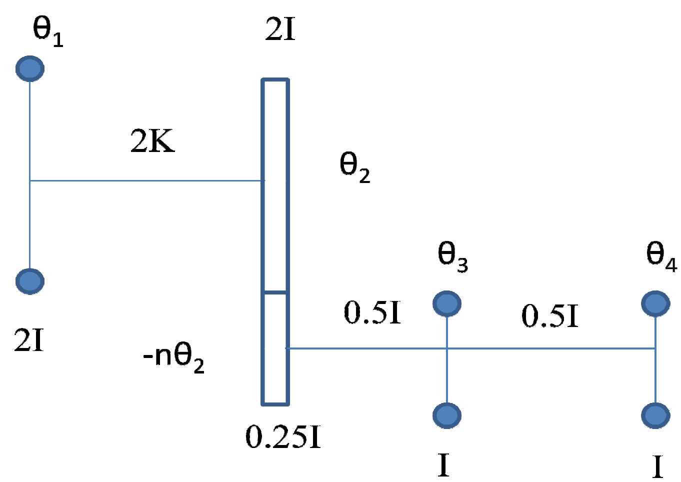


Fig. (4.1)

Input:

$I = 13.56$ kgm^2, $k = 0.407 * 10^6$ Nm/rad, n=2

Global Mass Matrix, [M] $= \begin{bmatrix} 27.12 & & & \\ & 40.68 & & \\ & & 13.56 & \\ & & & 13.56 \end{bmatrix}$

Element 1

$$[k^1] = \begin{bmatrix} 2k & -2k \\ -2k & 2k \end{bmatrix}$$

Element 2

$$[k^2] = \begin{bmatrix} 2k & k \\ k & \dfrac{k}{2} \end{bmatrix}$$

Element 3

$$[k^3] = \begin{bmatrix} \dfrac{k}{2} & \dfrac{-k}{2} \\ \dfrac{-k}{2} & \dfrac{k}{2} \end{bmatrix}$$

Global Stiffness Matrix,

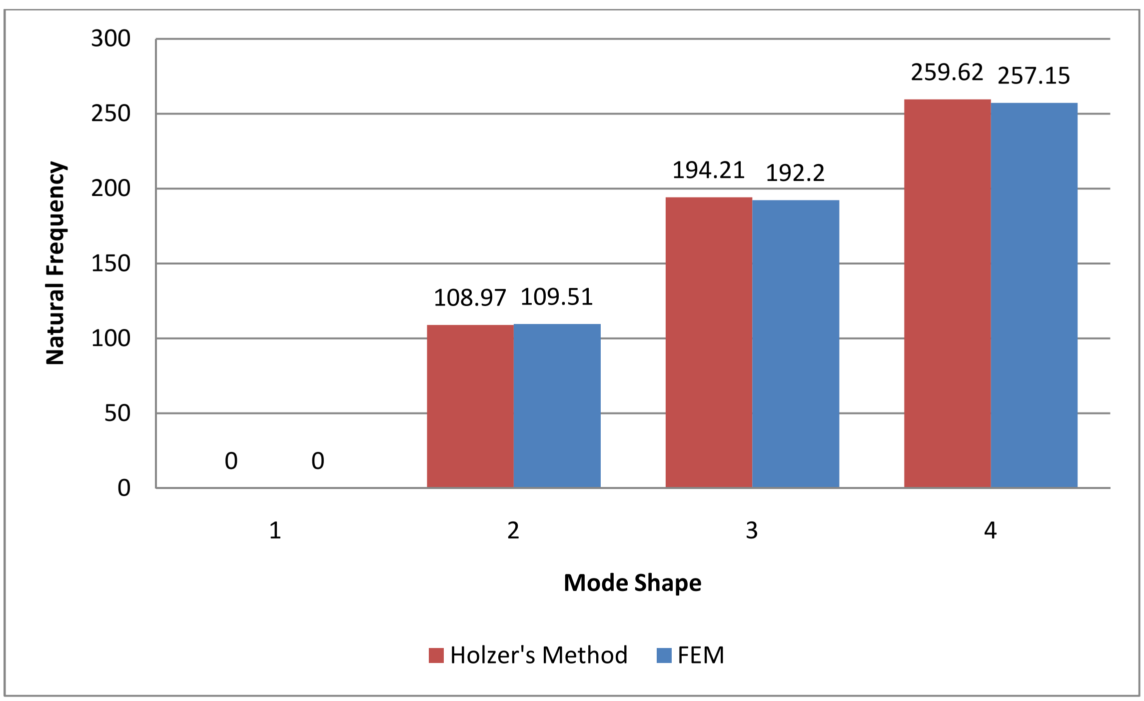
$$[K] = \begin{bmatrix} 812000 & -812000 & & \\ -812000 & 1624000 & 406000 & \\ & 406000 & 406000 & -203000 \\ & & -203000 & 203000 \end{bmatrix}$$

Eigen value equation,

$$\left| k - \omega^2 M \right| = 0$$

Solution:

Natural Frequencies (FEM) (rad/s)	0	109.51	192.20	257.15
Natural Frequencies (Holzer)	0	108.97	194.21	259.62
Difference	0	0.54	2.01	2.47
Eigen Vectors	1	1	1	1
	1.0000	0.6004	-0.2307	-1.2031
	-2.0000	0.3180	2.0710	-1.1390
	-2.0000	1.5827	-1.4171	0.3344



Comparison between Holzer’s Method and FEM

4.2 Three Branched Gear System

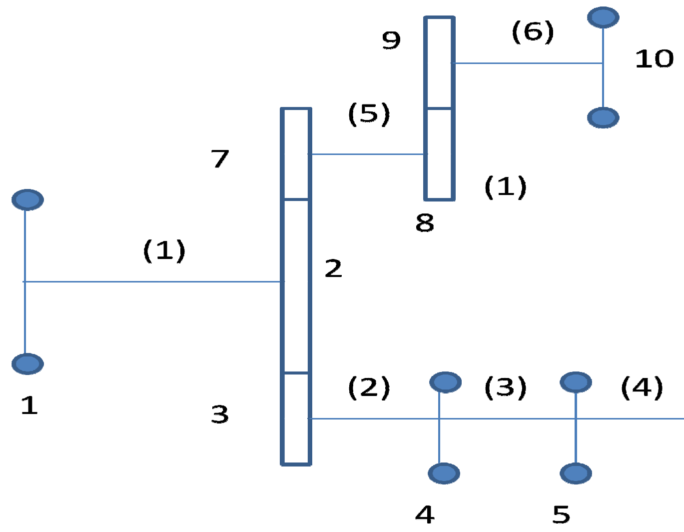


Fig. (4.2)

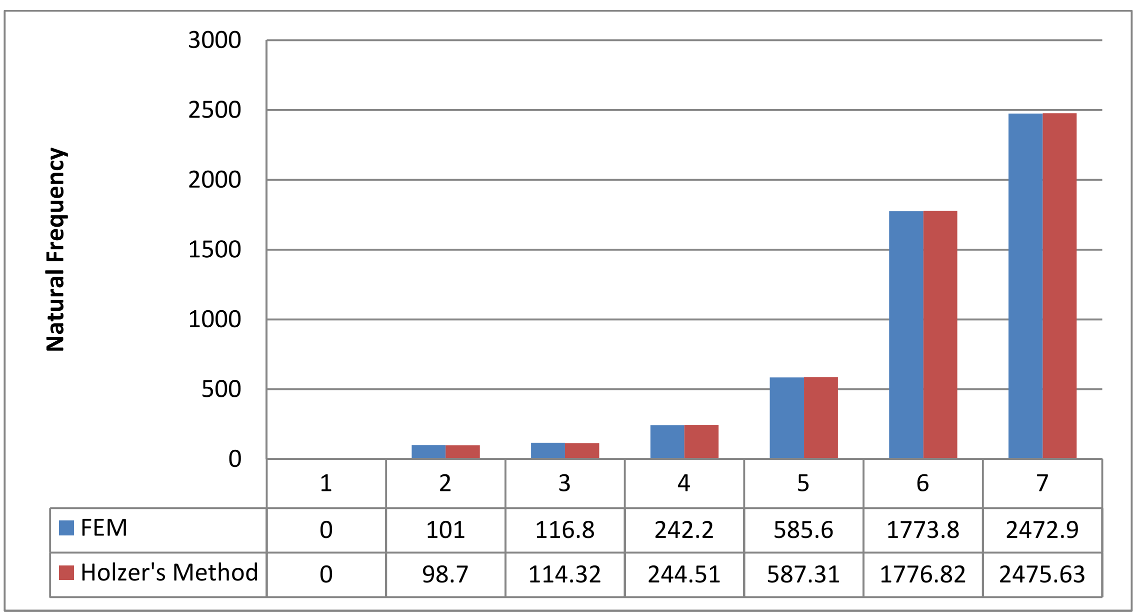
Data:

$n_1=2$, $n_2=4$, $n_3=8$

Gears (Rotors)		Shafts	
No.	I	No.	K(*10^6)
1	9700	1	130
2	980	2	166
3	36	3	8.2
4	400	4	4.2
5	230	5	150
6	230	6	22
7	3		
8	300		
9	0.3		
10	82		

Solution:

Natural Frequencies (FEM) (rad/s)	0	101	116.8	242.2	585.6	1773.8	2472.9
Natural Frequencies (Holzer)	0	98.7	114.32	244.51	587.31	1776.82	2475.63
Difference	0	2.3	2.48	2.31	1.71	3.02	2.73
Eigen Vectors	1	1	1	1	1	1	1
	1	2	0	-3.4	-24.7907	-235.6357	-458.9
	2.3	1.4	0.1	53.7	254.9616	-83.2125	-77.4
	-2.3	39.4	-0.1	827.5	-30.8929	0.9418	0.4
	-2.3	91.2	-0.2	365.7	1.7093	-0.0054	0
	-4.8	-2	0.3	-7.7	3.1860	210.7260	-2033.1
	40.9	17.7	-2.8	85.1	74.2991	155.3684	-740.1



Comparison between Holzer's Method and FEM

4.3 Six Branched Gear System

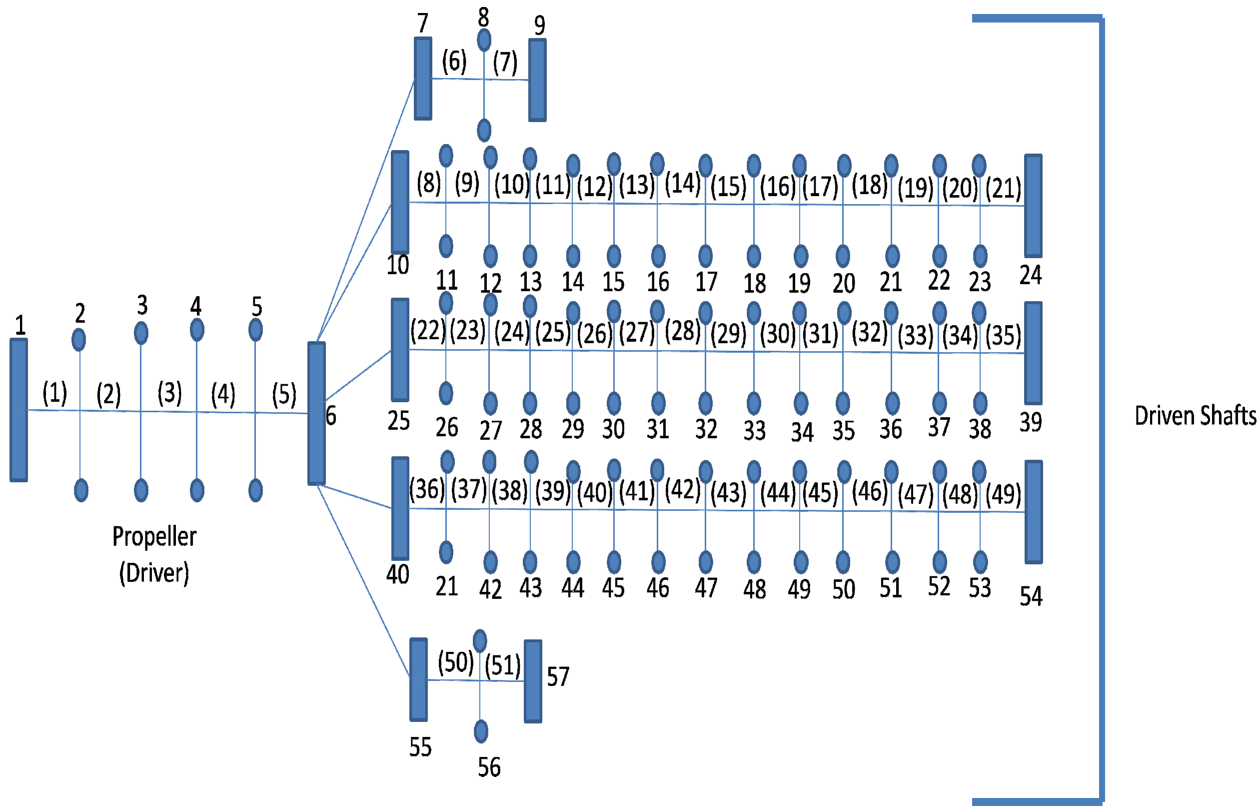


Fig. (4.3)

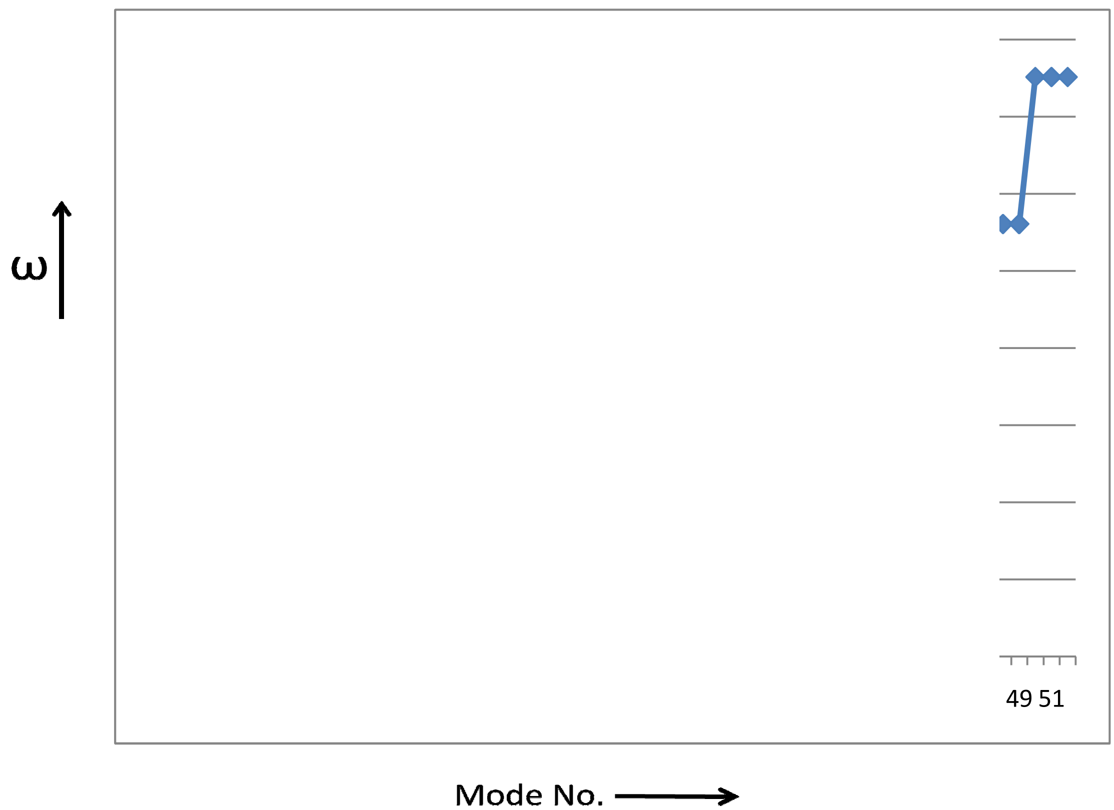
<u>Data:</u>

Gears (Rotors)		**Shafts**	
No.	*I*	*No.*	*K (*10^7)*
1	45000	1	21
2	4100	2	47
3	1470	3	11
4	670	4	7
5	370	5	61
6	3000	6	37
7	300	7	1
8	40	8	60
9	1400	9	175
10	300	10	2
11	270	11	110
12	16	12	190
13	110	13	140

14	1480	14	140
15	80	15	140
16	80	16	140
17	80	17	140
18	80	18	140
19	80	19	140
20	80	20	100
21	80	21	3
22	80		
23	9.8		
24	180		

$n_1=n_2=5$

First five Natural Frequencies (rad/s): **0, 27, 104, 116, 216**



Matalb Code Block:

```
% Defining Mass Matrix
mass =
diag([48000,4000,1478,672,374,3040+2*25*300+3*25*300,38.56,1408,272.8,16.4,10
8.4,1483.6,80.32,80.32,80.32,80.32,80.32,80.32,80.32,80.32,9.8,183.6,272.8,16
.4,108.4,1483.6,80.32,80.32,80.32,80.32,80.32,80.32,80.32,80.32,9.8,183.6,272
.8,16.4,108.4,1483.6,80.32,80.32,80.32,80.32,80.32,80.32,80.32,80.32,9.8,183.
6,38.56,1408]);

% Defining elemental Stiffness Matrix
k1=10^7*[20.964 -20.964;-20.964 20.964];
k2=10^7*[47.393 -47.393;-47.393 47.393];         %Propeller
k3=10^7*[10.97 -10.97;-10.97 10.97];
k4=10^7*[7.57 -7.57;-7.57 7.57];
k5=10^7*[60.967 -60.967;-60.967 60.967];

k6=10^7*[36.765*25 36.765*5;36.765*5 36.765];
k7=10^7*[1.575 -1.575;-1.575 1.575];           %Generator 1

k8=10^7*[58.14*25 58.14*5;58.14*5 58.14];
k9=10^7*[175.439 -175.439;-175.439 175.439];
k10=10^7*[175.439 -175.439;-175.439 175.439];
k11=10^7*[107.527 -107.527;-107.527 107.527];
k12=10^7*[185.185 -185.185;-185.185 185.185];
k13=10^7*[142.857 -142.857;-142.857 142.857];
k14=10^7*[142.857 -142.857;-142.857 142.857];  %Engine 1
k15=10^7*[142.857 -142.857;-142.857 142.857];
k16=10^7*[142.857 -142.857;-142.857 142.857];
k17=10^7*[142.857 -142.857;-142.857 142.857];
k18=10^7*[142.857 -142.857;-142.857 142.857];
k19=10^7*[142.857 -142.857;-142.857 142.857];
k20=10^7*[105.263 -105.263;-105.263 105.263];
k21=10^7*[2.959 -2.959;-2.959 2.959];

k22=10^7*[58.14*25 58.14*5;58.14*5 58.14];
k23=10^7*[175.439 -175.439;-175.439 175.439];
k24=10^7*[175.439 -175.439;-175.439 175.439];
k25=10^7*[107.527 -107.527;-107.527 107.527];
k26=10^7*[185.185 -185.185;-185.185 185.185];
k27=10^7*[142.857 -142.857;-142.857 142.857];
k28=10^7*[142.857 -142.857;-142.857 142.857];   %Engine 2
k29=10^7*[142.857 -142.857;-142.857 142.857];
k30=10^7*[142.857 -142.857;-142.857 142.857];
k31=10^7*[142.857 -142.857;-142.857 142.857];
k32=10^7*[142.857 -142.857;-142.857 142.857];
k33=10^7*[142.857 -142.857;-142.857 142.857];
k34=10^7*[105.263 -105.263;-105.263 105.263];
k35=10^7*[2.959 -2.959;-2.959 2.959];

k36=10^7*[58.14*25 58.14*5;58.14*5 58.14];
k37=10^7*[175.439 -175.439;-175.439 175.439];
k38=10^7*[175.439 -175.439;-175.439 175.439];
```

```
k39=10^7*[107.527 -107.527;-107.527 107.527];
k40=10^7*[185.185 -185.185;-185.185 185.185];
k41=10^7*[142.857 -142.857;-142.857 142.857];
k42=10^7*[142.857 -142.857;-142.857 142.857];  %Engine 3
k43=10^7*[142.857 -142.857;-142.857 142.857];
k44=10^7*[142.857 -142.857;-142.857 142.857];
k45=10^7*[142.857 -142.857;-142.857 142.857];
k46=10^7*[142.857 -142.857;-142.857 142.857];
k47=10^7*[142.857 -142.857;-142.857 142.857];
k48=10^7*[105.263 -105.263;-105.263 105.263];
k49=10^7*[2.959 -2.959;-2.959 2.959];

k50=10^7*[36.765*25 36.765*5;36.765*5 36.765];
k51=10^7*[1.575 -1.575;-1.575 1.575];         %Generator 2

stiff=zeros(52,52);

%Forming the global stiffness matrix using the function 'toglobal'
stiff=toglobal(stiff,k1,1,2);
stiff=toglobal(stiff,k2,2,3);
stiff=toglobal(stiff,k3,3,4);
stiff=toglobal(stiff,k4,4,5);
stiff=toglobal(stiff,k5,5,6);

stiff=toglobal(stiff,k6,6,7);
stiff=toglobal(stiff,k7,7,8);

stiff=toglobal(stiff,k8,6,9);
stiff=toglobal(stiff,k9,9,10);
stiff=toglobal(stiff,k10,10,11);
stiff=toglobal(stiff,k11,11,12);
stiff=toglobal(stiff,k12,12,13);
stiff=toglobal(stiff,k13,13,14);
stiff=toglobal(stiff,k14,14,15);
stiff=toglobal(stiff,k15,15,16);
stiff=toglobal(stiff,k16,16,17);
stiff=toglobal(stiff,k17,17,18);
stiff=toglobal(stiff,k18,18,19);
stiff=toglobal(stiff,k19,19,20);
stiff=toglobal(stiff,k20,20,21);
stiff=toglobal(stiff,k21,21,22);

stiff=toglobal(stiff,k22,6,23);
stiff=toglobal(stiff,k23,23,24);
stiff=toglobal(stiff,k24,24,25);
stiff=toglobal(stiff,k25,25,26);
stiff=toglobal(stiff,k26,26,27);
stiff=toglobal(stiff,k27,27,28);
stiff=toglobal(stiff,k28,28,29);
stiff=toglobal(stiff,k29,29,30);
stiff=toglobal(stiff,k30,30,31);
stiff=toglobal(stiff,k31,31,32);
stiff=toglobal(stiff,k32,32,33);
stiff=toglobal(stiff,k33,33,34);
stiff=toglobal(stiff,k34,34,35);
stiff=toglobal(stiff,k35,35,36);
```

```
stiff=toglobal(stiff,k36,6,37);
stiff=toglobal(stiff,k37,37,38);
stiff=toglobal(stiff,k38,38,39);
stiff=toglobal(stiff,k39,39,40);
stiff=toglobal(stiff,k40,40,41);
stiff=toglobal(stiff,k41,41,42);
stiff=toglobal(stiff,k42,42,43);
stiff=toglobal(stiff,k43,43,44);
stiff=toglobal(stiff,k44,44,45);
stiff=toglobal(stiff,k45,45,46);
stiff=toglobal(stiff,k46,46,47);
stiff=toglobal(stiff,k47,47,48);
stiff=toglobal(stiff,k48,48,49);
stiff=toglobal(stiff,k49,49,50);

stiff=toglobal(stiff,k50,6,51);
stiff=toglobal(stiff,k51,51,52);

%Eigen value equation is solved
[mode,freq]=eig(stiff,mass);

%Natural Frequencie are displayed
sort(sqrt(diag(freq)))

for i=1:length(mode)
    x=mode(:,i);
    Q(1,i)=1;
    for j=2:length(x)
        x(j)=x(j)/x(1);
        Q(j,i)=x(j);
    end

end

%The eigen vetors are dsplayed
Q
```

The Function ‘toglobal’

```
function y = toglobal(stiff,k,i,j)
%LinearBarAssemble   This function assembles the elemental stiffness
%                    matrix k's of the shafts in the geared system with nodes
%                    i and j into the global stiffness matrix stiff.
stiff(i,i) = stiff(i,i) + k(1,1);
stiff(i,j) = stiff(i,j) + k(1,2);
stiff(j,i) = stiff(j,i) + k(2,1);
stiff(j,j) = stiff(j,j) + k(2,2);
y = stiff;
```

Conclusion

The importance of studying torsional vibration in geared systems is studied. The various methods to determine the natural frequencies and mode shapes were mentioned, and especially for this project the classical numerical method of Holzer was pitched against the Finite Element Method. The Holzer's method, being an iterative process starting with an assumed value of natural frequency, is very laborious and involves a lot of time. On the other hand Finite Element Method provides a general formulation for the elemental stiffness and mass matrices of a shaft element. According to this method, the property matrices for the shaft element connecting with the slave gear can be determined by just incorporating the gear (or speed) ratio in the elemental matrices. Basically, branched geared systems are converted to a straight geared system. The point of gear meshing is considered as a single node. Here, three types of geared systems are studied (single, three and six branched). The natural frequencies obtained by both Finite Element Method and Holzer's method are calculated for the first two systems. And the results obtained by it are in good agreement with the FEM results. Also the discrete mode values are found out. The graphs depicting the natural frequencies for the various modes are also shown. Overall the project provides an insight into the effectiveness of the Finite Element Method in an important engineering problem of torsional vibration.

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